

Chapter 3

Static Equilibrium

3.1 The Important Stuff

In this chapter we study a *special case* of the dynamics of rigid objects covered in the last two chapters. It is the (very important!) special case where the center of mass of the object has no motion and the object is not rotating.

3.1.1 Conditions for Equilibrium of a Rigid Object

For a rigid object which is not moving *at all* we have the following conditions:

- The (vector) sum of the external forces on the rigid object must equal zero:

$$\sum \mathbf{F} = 0 \tag{3.1}$$

When this condition is satisfied we say that the object is in **translational equilibrium**. (It really only tells us that \mathbf{a}_{CM} is zero, but of course that includes the case where the object is motionless.)

- The sum of the external torques on the rigid object must equal zero.

$$\sum \boldsymbol{\tau} = 0 \tag{3.2}$$

When this condition is satisfied we say that the object is in **rotational equilibrium**. (It really only tells us that α about the given axis is zero, but —again— that includes the case where the object is motionless.)

When both 3.1 and 3.2 are satisfied we say that the object is in **static equilibrium**.

Nearly all of the problems we will solve in this chapter are two-dimensional problems (in the xy plane), and for these, Eqs. 3.1 and 3.2 reduce to

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum \tau_z = 0 \tag{3.3}$$

3.1.2 Two Important Facts for Working Statics Problems

i) The force of gravity acts on all massive objects in our statics problems; it acts on all the individual mass points of the object. One can show that for the purposes of computing the forces and torques on rigid objects in statics problems we can treat the mass of the entire object as being concentrated at its center of mass; that is, for an object of mass M we can treat gravity as exerting a force Mg downward at the center of mass.

(This result depends on the fact that the acceleration of gravity, \mathbf{g} is usually constant over the volume of the object. Otherwise it is not true.)

ii) While there is only one way to write the conditions for the *forces* on a rigid object summing to zero, we have a choice in the way we write the equation for the total torque. Eq. 3.3 does not specify the choice of the *axis* for calculating the torque. In general it matters a great deal which axis we pick! But when the sum of torques about any one axis is zero *and* the sum of forces is zero (translational equilibrium) then the sum of torques about *any* axis will give zero; so for statics problems we are free to pick the most convenient axis for computing $\sum \tau$. Often this will be the point on the object where several unknown forces are acting, so that the resulting set of equations will be simpler to solve.

3.1.3 Examples of Rigid Objects in Static Equilibrium

Strategy for solving problems in static equilibrium:

- Determine *all* the forces that are acting on the rigid body. They will come from the other objects with which the body is in contact (supports, walls, floors, weights resting on them) as well as gravity,
- Draw a diagram and put in *all* the information you have about these forces: The points on the body at which they act, their magnitudes (if known), their directions (if known).
- Write down the equations for static equilibrium. For the torque equation you will have a choice of where to put the axis; in making your choice think of which point would make the resulting equations the simplest.
- Solve the equations! (That's not physics. . . that's math.) If the problem is well-posed you will not have too many or too few equations to find all the unknowns.

3.2 Worked Examples

3.2.1 Examples of Rigid Objects in Static Equilibrium

1. The system in Fig. 3.1 is in equilibrium with the string in the center exactly horizontal. Find (a) tension T_1 , (b) tension T_2 , (c) tension T_3 and (d) angle θ .

[HRW5 13-23]

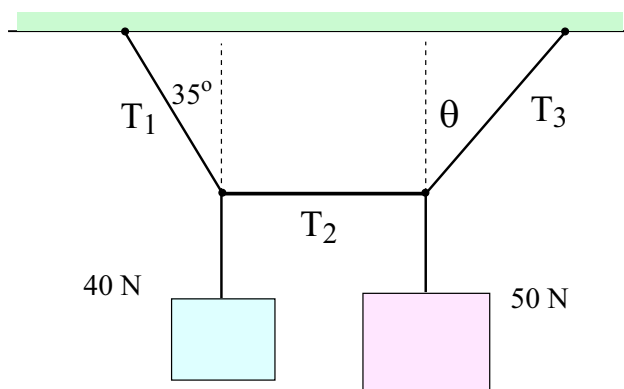


Figure 3.1: System of masses and strings for Example 1.

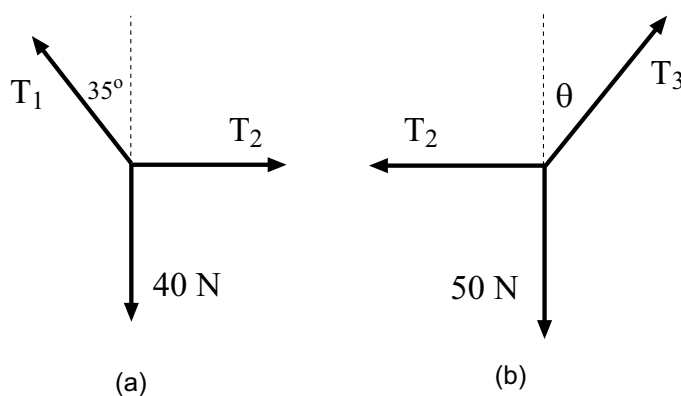


Figure 3.2: (a) Forces at the left junction of the strings. (b) Forces acting at the right junction of the strings.

Whoa! *Four* unknowns (T_1 , T_2 , T_3 and θ) to solve for! How will we ever figure this out?

We consider the points where the strings meet; the left junction is shown in Fig. 3.2 (a). Since a string under tension pulls inward along its length with a force given by the string tension, the forces acting at this point are as shown.

Since this junction in the strings is in static equilibrium, the (vector) sum of the forces acting on it must give zero. Thus the sum of the x components of the forces is zero:

$$-T_1 \sin 35^\circ + T_2 = 0 \quad (3.4)$$

and the sum of the y components of the forces is zero:

$$+T_1 \cos 35^\circ - 40 \text{ N} = 0 \quad (3.5)$$

Now we look at the right junction of the strings; the forces acting here are shown in Fig. 3.2 (b). Again, the sum of the x components of the forces is zero:

$$-T_2 + T_3 \sin \theta = 0 \quad (3.6)$$

and the sum of the y components of the forces is zero:

$$+T_3 \cos \theta - 50 \text{ N} = 0 \quad (3.7)$$

And at this point we are done with the *physics* because we have four equations for four unknowns. We will do algebra to solve for them.

In this problem the algebra really isn't so bad. From Eq. 3.5 we get

$$T_1 = \frac{(40 \text{ N})}{(\cos 35^\circ)} = 48.8 \text{ N}$$

and then Eq. 3.4 gives us T_2 :

$$T_2 = T_1 \sin 35^\circ = (48.8 \text{ N}) \sin 35^\circ = 28.0 \text{ N} .$$

We now rewrite Eq. 3.6 as:

$$T_3 \sin \theta = T_2 = 28.0 \text{ N} \quad (3.8)$$

and Eq. 3.7 as:

$$T_3 \cos \theta = 50.0 \text{ N} \quad (3.9)$$

Now if we divide the left and right sides of 3.8 by the left and right sides of 3.9 we get:

$$\tan \theta = \frac{(28.0 \text{ N})}{(50.0 \text{ N})} = 0.560$$

and then

$$\theta = \tan^{-1}(0.560) = 29.3^\circ$$

Finally, we get T_3 from Eq. 3.9:

$$T_3 = \frac{(50.0 \text{ N})}{(\cos 29.3^\circ)} = 57.3 \text{ N}$$

Summarizing, we have found:

$$T_1 = 48.8 \text{ N} \quad T_2 = 28.0 \text{ N} \quad T_3 = 57.3 \text{ N} \quad \theta = 29.3^\circ$$

This answers all the parts of the problem.

2. The system in Fig. 3.3 is in equilibrium. A mass of 225 kg hangs from the end of the uniform strut whose mass is 45.0 kg. Find (a) the tension T in the cable and the (b) horizontal and (c) vertical force components exerted on the strut by the hinge. [HRW5 13-33]

(a) The rigid body here is the strut. What are the forces acting on it?

We know the mass M of the strut; the force of gravity exerts a force Mg downward at its center of mass (which is in the middle of the strut since it is uniform). If the hanging mass is $m = 225 \text{ kg}$ then the string which supports it exerts a downward force of magnitude mg at the top end of the strut. The cable attached to the top of the strut exerts a force of magnitude T . What is its direction? Some geometry (shown in Fig. 3.4) shows that its

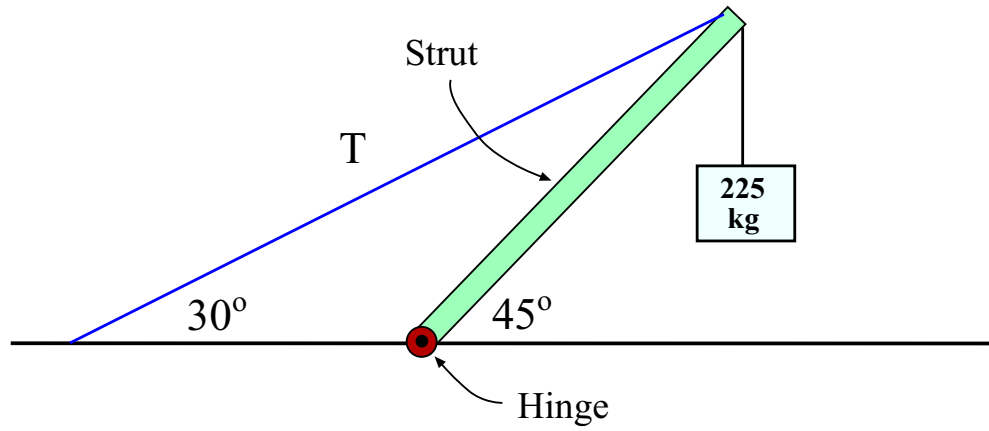


Figure 3.3: Geometry of the statics problem of Example 2.

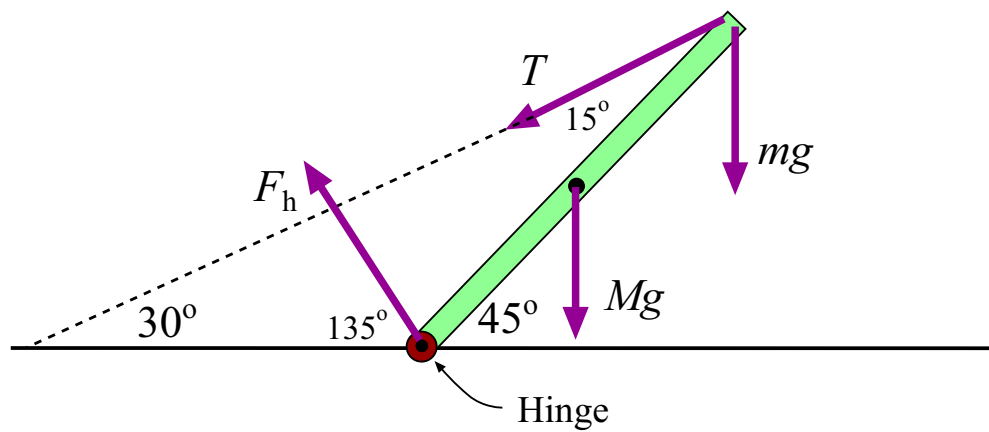


Figure 3.4: Forces acting on the strut in Example 2.

direction makes an angle of 15° with the strut. Finally the *hinge* exerts a force on the strut. (Can't forget that... the hinge is in contact with the metal bar which is the "strut" as so exerts a force on it.) The magnitude of this force is just labelled F_h in the diagram, but we don't know its direction!

Now, one way to solve the problem would be to let the direction of the hinge force be some angle θ as measured from some line of reference. In fact it will probably be easiest to let the x and y components of this force be the unknowns... I will call them $F_{h,x}$ and $F_{h,y}$. In fact, parts (b) and (c) of the problem ask us for these components directly. We can always get the direction and magnitude later!

Now let's write down some equations. First, the sum of the F_x 's must give zero. Note (from basic geometry) that the force of the cable is directed at 30° below the horizontal. And force of the hinge has an x component! Then from Fig. 3.4 we immediately read off:

$$F_{h,x} - T \cos 30^\circ = 0 \quad (3.10)$$

Good enough. Now onto the F_y equation. The sum of the y components of the forces gives zero, and we write:

$$F_{h,y} - T \sin 30^\circ - Mg - mg = 0 \quad (3.11)$$

Now we use the condition for zero net torque. The question is: Where do we want to put the axis? For this problem, the answer is obvious. We want to put it at the hinge itself because then when we calculate the torques, the hinge force (with its two unknown components) will give *no* torque. The equation will still be useful... and it will be much simpler. (Keep in mind that even though a physical strut really does turn around a physical hinge we still have the *choice* of putting the axis for torque anywhere.)

We are not told the length of the strut, so let its length be L . We note the angles that the force vectors make with the line joining the axis to the points of application, and then we write the sum of the torques as:

$$-Mg \frac{L}{2} \sin 45^\circ - mgL \sin 45^\circ + TL \sin 15^\circ = 0$$

but we note that we can cancel the L out of this equation, leaving

$$-\frac{Mg}{2} \sin 45^\circ - mg \sin 45^\circ + T \sin 15^\circ = 0 \quad (3.12)$$

Are we done with the *physics* yet? In the Eqs. 3.10, 3.11 and 3.12 there are three unknowns: T , $F_{h,x}$ and $F_{h,y}$. We *are* done with the physics. Only algebra remains.

And the algebra isn't so bad. The *only* unknown in Eq. 3.12 is T and we get:

$$\begin{aligned} T \sin 15^\circ &= \frac{Mg}{2} \sin 45^\circ + mg \sin 45^\circ \\ &= \frac{1}{2}(45 \text{ kg})(9.80 \frac{\text{m}}{\text{s}^2}) \sin 45^\circ + (225 \text{ kg})(9.80 \frac{\text{m}}{\text{s}^2}) \sin 45^\circ \\ &= 1715 \text{ N} \end{aligned}$$

so that

$$T = \frac{(1715 \text{ N})}{\sin 15^\circ} = 6.63 \times 10^3 \text{ N}$$

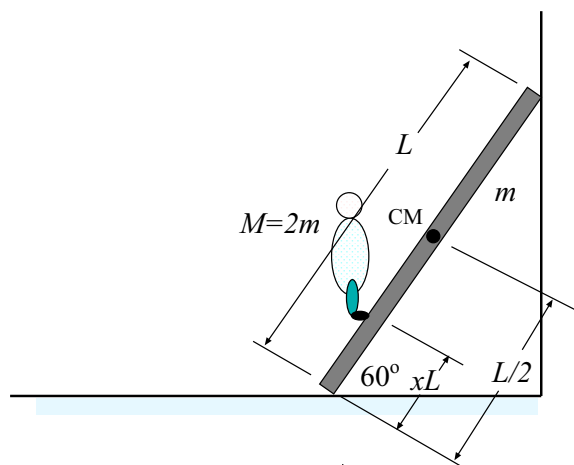


Figure 3.5: Geometry of the statics problem of Example 3. Student is standing on a ladder which leans against a wall.

(b) With our result from part(a) in hand, $F_{h,x}$ and $F_{h,y}$ will be easy to find. From 3.10 we get:

$$F_{h,x} = T \cos 30^\circ = (6630 \text{ N}) \cos 30^\circ = 5.74 \times 10^3 \text{ N}$$

and From 3.11 we get:

$$\begin{aligned} F_{h,y} &= T \sin 30^\circ + Mg + mg \\ &= (6630 \text{ N}) \sin 30^\circ + (45 \text{ kg})(9.80 \frac{\text{m}}{\text{s}^2}) + (225 \text{ kg})(9.80 \frac{\text{m}}{\text{s}^2}) \\ &= 5.96 \times 10^3 \text{ N} \end{aligned}$$

The horizontal and vertical components of the force of the hinge on the strut are

$$F_{h,x} = 5740 \text{ N} \quad F_{h,y} = 5960 \text{ N} .$$

3. A ladder having a uniform density and a mass m rests against a frictionless vertical wall at an angle of 60° . The lower end rests on a flat surface where the coefficient of static friction is $\mu_s = 0.40$. A student with a mass $M = 2m$ attempts to climb the ladder. What fraction of the length L of the ladder will the student have reached when the ladder begins to slip? [Ser4 12-13]

We make a basic diagram of the geometry of the problem in Fig. 3.5. The ladder has length L ; we show the center of mass of the ladder at a distance $\frac{L}{2}$ up from its bottom end. If the student had climbed a fraction x of the ladder, then he/she is at a distance xL from its lower end, as shown.

Keeping the geometry in mind, we next think about *all* the separate forces that are acting on the ladder as it leans against the wall and supports the student.

The force of gravity mg (downward) is effectively exerted at the center of the ladder. Since the ladder is exerting an upward force Mg on the student, the student must be exerting a

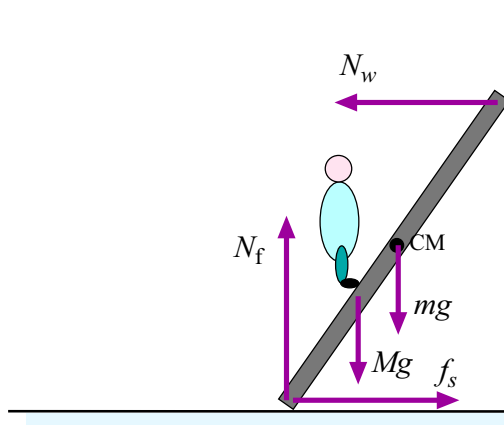


Figure 3.6: Forces acting on the ladder in Example 3. Many TTU students *do* look like this guy.

downward force of magnitude $Mg = 2mg$ on the ladder at a point a distance xL from the lower end. The wall is frictionless so it can only exert a *normal* force on the top end of the ladder; we will denote the magnitude of this force by N_w . The floor will exert a normal force N_f upward on the bottom end of the ladder but also a horizontal force of static friction. Which way does this friction force point? In our diagram, the wall's normal force points to the left so the friction force *must* point to the right so that the forces can add up to zero.

All these forces and their directions are diagrammed in Fig. 3.6. Now we apply the conditions for static equilibrium given in Eq. 3.3.

First off, the horizontal forces must sum to zero. That gives us:

$$f_s - N_w = 0 \quad (3.13)$$

Next, the vertical forces must sum to zero. This gives:

$$N_f - Mg - mg = 0$$

and using $M = 2m$ we get:

$$N_f - 3mg = 0 \quad (3.14)$$

or: $N_f = 3mg$, giving us an expression for the normal force of the floor.

The next condition for equilibrium is that the sum of torques *taken about any axis* must give zero. Since we have two forces acting at the lower end of the ladder, it might be best to put the axis there because then those forces will give no torque, and we will have a simpler equation to deal with. We note that the gravity forces from the student and the ladder's CM make an angle of 30° with the line joining the axis to the application points; they give a clockwise (negative) torque. The normal force from the wall makes an angle of 60° with the line from the axis, and it gives a positive torque. Our equation is:

$$-(xL)(2mg) \sin 30^\circ - (L/2)(mg) \sin 30^\circ + (L)(N_w) \sin 60^\circ = 0$$

This equation can be simplified: We can cancel an L and use the values of $\sin 30^\circ$ and $\sin 60^\circ$ to get:

$$-xmg - \frac{mg}{4} + \frac{\sqrt{3}}{2}N_w = 0 \quad (3.15)$$

We now have *three* equations, with *four* unknowns N_f , N_w , f_s and x . (We can consider m as given. In fact, its value won't matter.) We need another equation!

We have not yet used the condition that when the student is standing xL up from the ladder's bottom *it is just about to slip*. Why should the ladder slip at all? It is because the force of static friction f_s is limited in size; we know that it can only be as large as $\mu_s N_f$, since N_f is the normal force between the floor and the ladder's lower end. When the student has walked up far enough that the ladder is on the verge of slipping then we have the equality

$$f_s = \mu_s N_f \quad (3.16)$$

That's all the equations! Now let's start solving them. (The *physics* is done. The *math* remains.)

Using Eq. 3.16 in Eq. 3.13 gives

$$\mu_s N_f - N_w = 0$$

but from 3.14 we had $N_f = 3mg$, so we get

$$3\mu_s mg = N_w$$

Put this result in Eq. 3.15 and we have:

$$-xmg - \frac{mg}{4} + \frac{\sqrt{3}}{2}(3\mu_s mg) = 0$$

We can cancel out the factor mg since it appears in each term. (So we never needed to know m):

$$-x - \frac{1}{4} + \frac{3\sqrt{3}\mu_s}{2} = 0$$

And at last, using the given value $\mu_s = 0.400$:

$$x = \frac{3\sqrt{3}(0.400)}{2} - \frac{1}{4} = 0.798$$

The student can climb a fraction of 0.789 (that is, nearly 80%) of the length of the ladder before it starts to slip.

4. For the stepladder shown in Fig. 3.7, sides AC and CE are each 8.0ft long and hinged at C . Bar BD is a tie-rod 2.5ft long, halfway up. A man weighing 192lb climbs 6.0ft along the ladder. Assuming that the floor is frictionless and neglecting the weight of the ladder, find (a) the tension in the tie-rod and the forces exerted on the ladder by the floor at (b) A and (c) E . *Hint: It will help to isolate parts of the ladder in applying the equilibrium conditions.* [HRW5 13-41]

Before anything else, we need to do a little geometry to find the angles which the ladder sides make with the floor, because we *are* given enough information for this in the problem.

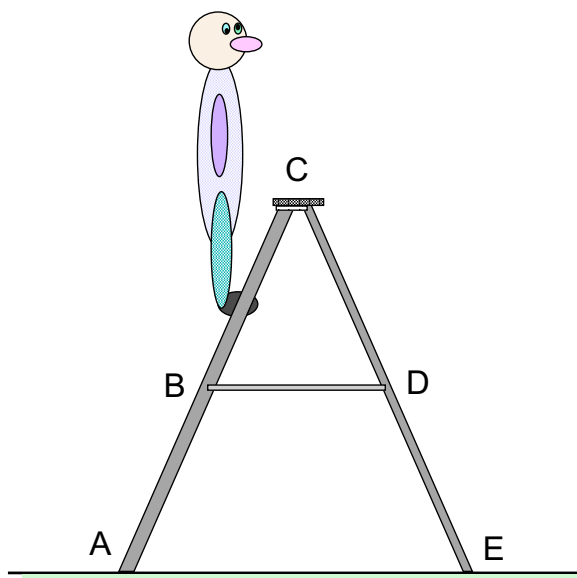


Figure 3.7: Man stands on a ladder in Example 4.

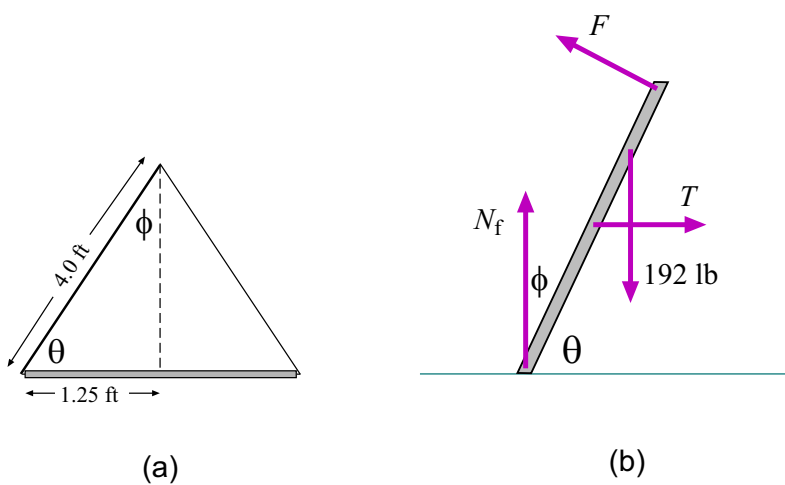


Figure 3.8: (a) Some trigonometry to find the angle of slope of the ladder. (b) The forces which act on the left side of the ladder and their application points.

We note that since the sides are of equal length then triangle ACE is an isosceles triangle. The part of the ladder above the tie-rod has the lengths shown in Fig. 3.8(a). From this we can see that the angle which either side of the ladder makes with the ground is the same as the angle θ shown in the figure, and θ is given by

$$\cos \theta = \frac{1.25}{4.00} = 0.3125 \quad \implies \quad \theta = 71.79^\circ$$

and the angle ϕ (which we'll also need) is the complement of θ and is given by

$$\phi = 90^\circ - \theta = 18.21^\circ$$

Now that the geometry is settled, we follow the hint and treat the sides of the ladder as *separate objects* for which we will apply the conditions of static equilibrium. First, consider the left side of the ladder (the one on which the man is standing). What forces are acting on it?

The floor is frictionless (!) and so it can only exert a normal (vertical) force on the lower end. We will let this force be N_f . Moving up, the tie-bar is under a tension T , so it exerts a force of magnitude T which (we would guess) is pulling inward along its length, and so points to the right. This force is applied at the midpoint of the side. The 192 lb man is supported by the ladder, so he must be exerting a *downward* force of 192 lb at the point that is 6 ft up from the lower end.

But we're not done. At point C , the *right side* of the ladder is exerting a force on the *left side* (the one we are now considering). This is true because the two parts of the ladder are in contact at this point. Right now it is not at all clear which direction this force will point, so we will just say it is *some* force \mathbf{F} with components F_x and F_y which we will determine in the course of solving the equations. This force is indicated in the figure. *Now* we have all the forces acting on the left side of the ladder; they are diagrammed in Fig. 3.8 (b).

Thinking ahead to when we do our analysis for the right side of the ladder... we note that from Newton's 3rd law the sides of the ladder will exert "equal and opposite" forces on each other at point C . So the force of the *left* side on the *right* side will be $-\mathbf{F}$. But that comes later.

The sum of the x forces on this part of the ladder gives zero. That gives us:

$$F_x + T = 0 \tag{3.17}$$

The sum of the y forces on this part of the ladder gives zero:

$$F_y - 192 \text{ lb} + N_f = 0 \tag{3.18}$$

Next, the sum of the torques on the left ladder section must give zero. We will choose the location for our axis to be at the *top* (that is, point C); the reason for this choice is that the unknown force \mathbf{F} (which acts at C) will then give no torque. Using the given dimensions of the ladder and the application points of the forces (as well as the angles θ and ϕ that we figured out), we find:

$$-(8 \text{ ft})N_f \sin \phi + (2 \text{ ft})(192 \text{ lb}) \sin \phi + (4 \text{ ft})T \sin \theta = 0 \tag{3.19}$$

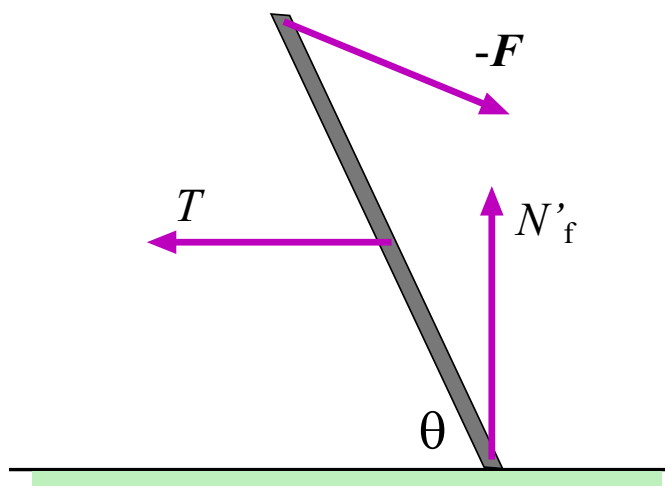


Figure 3.9: The forces which act on the right side of the ladder.

And now we analyze right side of the ladder. Starting from the bottom, we have the force of the floor. The floor is frictionless, so again the force points straight up, but it will not have the same magnitude as on the left side; we denote its magnitude here by N'_f . Then, midway up the length of the ladder the tension of the tie-rod pulls with a force of magnitude T to the left. Finally at point C , the *left* side of the ladder is exerting a force. But it must be opposite to the force we already assigned as coming from the right side. So the left side exerts a force $-\mathbf{F}$ on the right side of the ladder. These forces are diagrammed in Fig. 3.9.

And now we get the equations. The sum of the x forces on this part of the ladder gives zero:

$$-F_x - T = 0$$

But note that this equation is basically the same as Eq. 3.17. So we can ignore it!

The sum of the y forces on this part of the ladder gives zero:

$$-F_y + N'_f = 0 \quad (3.20)$$

Finally, the sum of the torques on the left ladder section must give zero. Again we choose the top (point C) as the location for our axis (for the same reasons as before). Using the given dimensions of the ladder and the application points of the forces we find:

$$-(4\text{ft})T \sin \theta + (8\text{ft})N'_f \sin \phi = 0 \quad (3.21)$$

And now we stop and rest and size up where we stand with the solution. We have found five equations, namely Eqs. 3.17 through 3.21. Our unknowns are N_f , N'_f , T , F_x and F_y . So we have enough equations to solve the problem. The *physics* is done. Only the *math* remains.

Here's *one* way to go about solving this set of equations.

Notice that F_x only appears in 3.17, so we can wait until the very end to use it (if we want to get F_x). Combining 3.18 and 3.20 we get

$$F_y = 192\text{lb} - N_f \quad \text{and} \quad F_y = N'_f$$

so

$$N'_f = 192 \text{ lb} - N_f$$

or

$$N_f = 192 \text{ lb} - N'_f \quad (3.22)$$

From Eq. 3.21, we cancel a factor of 4 ft and find

$$T = \frac{2N'_f \sin \phi}{\sin \theta} = 0.658N'_f \quad (3.23)$$

Now we use results 3.22 and 3.23 in Eq. 3.19 to substitute for N_f and T . First, in 3.19 we can cancel a factor of 2 ft and rearrange to get:

$$-4N_f \sin \phi + 2T \sin \theta = -(192 \text{ lb}) \sin \phi$$

Now substitute for N_f and T . This gives:

$$-4(192 \text{ lb} - N'_f) \sin \phi + 2(0.658)N'_f \sin \theta = -(192 \text{ lb}) \sin \phi$$

Some more regrouping and evaluation of terms gives

$$(2.50)N'_f = 180 \text{ lb}$$

and at last we have an answer:

$$N'_f = \frac{(180 \text{ lb})}{(2.50)} = 72 \text{ lb}$$

Having one solution, we can quickly get all the rest:

$$N_f = 192 \text{ lb} - N'_f = 120 \text{ lb}$$

$$T = \frac{2(72 \text{ lb}) \sin \phi}{\sin \theta} = 47 \text{ lb}$$

Glancing at what the problem *asked* us for, we see that we've now answered all the parts:

$$(a) \quad T = 47 \text{ lb} \quad (b) \quad N_f = 120 \text{ lb} \quad (c) \quad N'_f = 72 \text{ lb}$$

But being the thorough kind of guy that I am, I'd like to find the components of the force \mathbf{F} which the ladder parts exert on each other at C . They are:

$$F_x = -T = -47 \text{ lb} \quad F_y = N'_f = 72 \text{ lb}$$

(So the direction that I chose for \mathbf{F} rather arbitrarily in the figures was the correct one.)

